# Bolyai – Gerwien theorem for unbounded polygons

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**Abstract**

Using the notion of a limiting angle we prove a version of the classical Bolyai – Gerwien theorem for unbounded polygons on the Euclidean plane.

## 1 Introductions

The classical Bolyai – Gerwien theorem states the following:

**Bolyai – Gerwien Theorem.** *On the Euclidean plane any two simple polygons of equal area are scissor-congruent, that is one can be cut into finitely many polygonal pieces out of which one can reassemble the other polygon.*

Note that "reassembled" means that each piece can be moved by some isometry of the plane to form the polygon. The Bolyai – Gerwien Theorem was extensively studied, and was generalized to other geometries with suitable notion of reassemblance (see, for example, [1]).

A closed simply connected domain $\mathcal{P}$ on the plane is *an unbounded polygon* if its boundary $\partial\mathcal{P}$ consists of finite number of line segments and rays. We will always assume that at least one side of $\mathcal{P}$ contains a ray, that is we don't treat classical polygons in this note.

Since in the unbounded case polygons have infinite areas, the Bolyai – Gerwien theorem cannot be applied to determine scissor-congruence. In [2] the question regarding possible extensions of the classical result for such objects was posed. In the present note we answer this question.

## 2 Preliminaries and the main result

In order to state the main result, we need to introduce a useful notion due to A. D. Alexandrov (see [3]). Suppose $O$ is an arbitrary point on the plane. For any ray $l'$ starting at a point $O' \in \mathcal{P}$ and entirely lying inside or on the boundary of $\mathcal{P}$, let us consider a ray $l$ collinear to $l'$ and starting at $O$. The set of all such rays $l$ detached to a point $O$ we will call *the limiting angle of* $\mathcal{P}$. This angle does not depend on the choice of $O$, and will be denoted as $\Phi(\mathcal{P})$. Limiting angle can be defined in the same manner for any figure on the plane.

Observe that $\Phi(\mathcal{P})$ is not necessarily a solid plane angle, but may be a union of solid angles and rays having $O$ as a common vertex or a starting point. An opening at infinity of $\mathcal{P}$ that corresponds to a separate ray in $\Phi(\mathcal{P})$ is called a *parallel opening*. Obviously, we can calculate the width of a parallel opening at infinity as the distance between two parallel rays making it. Let us denote by $W(\mathcal{P})$ the sum of widths of all parallel openings at infinity of $\mathcal{P}$.

For two arbitrary (unbounded) polygons $\mathcal{P}$ and $\tilde{\mathcal{P}}$, we say that $|\Phi(\mathcal{P})| = |\Phi(\tilde{\mathcal{P}})|$ if the sum of angular measures of all solid angles making $\Phi(\mathcal{P})$ is equal to the similar quantity of $\Phi(\tilde{\mathcal{P}})$. Apparently, $|\Phi(\mathcal{P})| = 0$ if $\mathcal{P}$ posses only parallel openings at infinity.

Recall, that two plane polygons $\mathcal{P}_1$ and $\mathcal{P}_2$ (possibly unbounded) are called *scissor-congruent* or *equidecomposable* (denoted as $\mathcal{P}_1 \simeq \mathcal{P}_2$) if they can be cut into two equal finite sets of polygonal pieces. If the polygons are unbounded, we also allow unbounded polygons in these sets. Note that a piece of a polygon is considered to be a closed set, that is each piece is taken with its boundary.

Now we can state the main result – unbounded version of the Bolyai – Gerwien theorem.

**Theorem 1.** *Suppose $\mathcal{P}$ and $\tilde{\mathcal{P}}$ are two unbounded polygons on the Euclidean plane; then $\mathcal{P} \simeq \tilde{\mathcal{P}}$ if and only if one of the following conditions are met:*

1. $|\Phi(\mathcal{P})| = |\Phi(\tilde{\mathcal{P}})|$*, and both quantities are non-zero;*
2. $|\Phi(\mathcal{P})| = |\Phi(\tilde{\mathcal{P}})| = 0$*, and* $W(\mathcal{P}) = W(\tilde{\mathcal{P}})$*.*

# 3 Proof of the unbounded Boyai – Gerwien theorem

First, let us show the necessity part. In other words, we need to show that scissor-congruence preserves the value of the limiting angles and, in case they are zero, the width.

Assume that $\mathcal{P} \simeq \tilde{\mathcal{P}}$, and let $F \subseteq \mathcal{P}$ be any unbounded piece in the decomposition of $\mathcal{P}$. Then $F$ has an equal copy $\tilde{F} \subseteq \tilde{\mathcal{P}}$ in the decomposition of $\tilde{\mathcal{P}}$. Since $F$ and $\tilde{F}$ are both unbounded, they have non-trivial limiting angles $\Phi(F)$ and $\Phi(\tilde{F})$ (possibly, of zero values). At the same time, $F$ and $\tilde{F}$ differ only by an isometry of the plane, thus

$$|\Phi(F)| = |\Phi(\tilde{F})|. \tag{3.1}$$

Because $\mathcal{P}$ has no self-intersections, we get that if $F_i$ with $i \in \{1, \ldots, N\}$ are all unbounded pieces in the decomposition of $\mathcal{P}$, then

$$|\Phi(\mathcal{P})| = \sum_{i=1}^{N} |\Phi(F_i)|. \tag{3.2}$$

In other words, unbounded pieces from the decomposition don't have a common share in the limiting angle. Since $\mathcal{P}$ and $\tilde{\mathcal{P}}$ are equidecomposable, all $F_i$ have their isometrical copies $\tilde{F}_i$ in the decomposition of $\tilde{\mathcal{P}}$. Moreover, $\tilde{F}_i$ are the only unbounded pieces in the decomposition of $\tilde{\mathcal{P}}$. Therefore, (3.2) may be applied for $\tilde{\mathcal{P}}$ and $\tilde{F}_i$, and with (3.1) it will give the necessary condition $|\Phi(\mathcal{P})| = |\Phi(\tilde{\mathcal{P}})|$ (possibly, both equal to zero).

In order to finish the proof of the necessity part, we should show that from $\mathcal{P} \simeq \tilde{\mathcal{P}}$ and $|\Phi(\mathcal{P})| = |\Phi(\tilde{\mathcal{P}})| = 0$ it follows that $W(\mathcal{P}) = W(\tilde{\mathcal{P}})$. Indeed, vanishing values of polygons' limiting angles imply that both polygons contain only parallel openings at infinity. Let $F \subseteq \mathcal{P}$, $\tilde{F} \subseteq \tilde{\mathcal{P}}$ be two isometrical pieces from the decomposition of the polygons. Then since the polygons have only parallel openings at infinity, the only two unbounded edges of $F$ (and $\tilde{F}$) are parallel. Moreover,

$$W(F) = W(\tilde{F}). \tag{3.3}$$

Similarly to (3.1),

$$W(\mathcal{P}) = \sum_{i=1}^{N} W(F_i),\ W(\tilde{\mathcal{P}}) = \sum_{i=1}^{N} W(\tilde{F}_i) \tag{3.4}$$

where $F_i$ are all unbounded pieces of $\mathcal{P}$ with parallel openings at infinity, and $\tilde{F}_i$ are their isometric copies in the decomposition of $\tilde{\mathcal{P}}$. Combining (3.3) and (3.4), we obtain the necessity of condition 2 of Theorem 1.

The proof of sufficiency is constructive. The idea is to cut out infinite parts of polygons, while leaving finite parts of equal areas, and apply the classical Boyai – Gerwien theorem.

Suppose for an unbounded polygon $\mathcal{P}$ we have a decomposition $\mathcal{P} = \mathcal{G} \cup \mathcal{F}$, where $\mathcal{G}$ and $\mathcal{F}$ are the unions of all bounded and, respectively, unbounded pieces in the partition of $\mathcal{P}$. In particular, put $\mathcal{F} = \bigcup\limits_{i=1}^{N} F_i$ with $F_i$ being unbounded pieces. We will be looking for a decomposition whose pieces satisfy the following core assumptions:

1. For any $i_1$ and $i_2$, $F_{i_1} \cap F_{i_2} = \emptyset$.
2. For each $i$, the set $F_i \cap \mathcal{G}$ forms a segment $a_i$ of the length $|a_i|$. Note that from the first assumption, $F_i \cap \mathcal{G}$ is the set of all bounded edges of $F_i$, whereas $\partial F_i \backslash \mathcal{G}$ is a union of two rays, which we will denote $l_i$ and $r_i$.
3. If $F_i$ is not a parallel opening at infinity, then we assume that the bounded part $\mathcal{G}$ is such that $\angle(a_i, l_i) = \angle(a_i, r_i) = \varphi_i$ with $\varphi_i \in [\pi/2, \pi)$. Keeping the angle in the indicated range may require additional subdivisions of unbounded pieces along their inner rays. Note that $|\Phi(F_i)| = 2\varphi_i - \pi$.
4. If for some other polygon $\tilde{\mathcal{P}}$ we have a partition $\tilde{\mathcal{P}} = \tilde{\mathcal{G}} \cup \tilde{\mathcal{F}}$ into the bounded ($\tilde{\mathcal{G}}$) and the unbounded ($\tilde{\mathcal{F}}$) parts satisfying assumptions 1 – 3 above, then we assume that the areas of $\mathcal{G}$ and $\tilde{\mathcal{G}}$ are equal. By the classical Bolyai – Gerwien Theorem, this implies $\mathcal{G} \simeq \tilde{\mathcal{G}}$.

The assumptions above don't imply anything for pieces that are parallel openings at infinity. In each of the cases of our theorem we will treat such pieces differently.

First, let us settle case 2 of Theorem 1. According to this case, the infinite parts of any partition $\mathcal{P} = \mathcal{G} \bigcup_{i=1}^{N} F_i$ and $\tilde{\mathcal{P}} = \tilde{\mathcal{G}} \bigcup_{i=1}^{N} \tilde{F}_i$ contain only parallel openings at infinity $F_i$ and, respectively, $\tilde{F}_i$. Assume that all they satisfy assumption 3 above (despite being parallel openings). Since they are parallel, we will get $\varphi_i = \tilde{\varphi}_j = \pi/2$, where $\tilde{\varphi}_j$ are the similar to $\varphi_i$ angles defined for $\tilde{\mathcal{P}}$. Therefore, $W(F_i) = |a_i|$, and $W(\tilde{F}_i) = |\tilde{a}_i|$. Moreover, by condition 2 and using (3.4),

$$\sum_{i=1}^{N} |a_i| = W(\mathcal{P}) = W(\tilde{\mathcal{P}}) = \sum_{i=1}^{\tilde{N}} |\tilde{a}_i|.$$

The last equality tells us that subdividing pieces $F_i$ and $\tilde{F}_i$, if necessary, with rays parallel to the corresponding edges, we can ensure that $|a_i| = |\tilde{a}_i|$ for each $i$. But since for any $i$, $F_i$ and $\tilde{F}_i$ are parallel openings, the equality of their widths implies that the pieces are isometric. This, together with core assumption 4 on the finite parts $\mathcal{G}$, $\tilde{\mathcal{G}}$, gives us scissor-congruence of $\mathcal{P}$ and $\tilde{\mathcal{P}}$. Sufficiency of case 2 of Theorem 1 is thus proved.

Finally, let us stick to case 1 of the theorem. The condition $|\Phi(\mathcal{P})| = |\Phi(\tilde{\mathcal{P}})| \neq 0$ means that for any decompositions of $\mathcal{P}$ and $\tilde{\mathcal{P}}$ their infinite parts $\mathcal{F}$ and $\tilde{\mathcal{F}}$ contain at least one infinite piece that makes non-trivial contribution into the limiting angle. All such pieces we will denote, as usual, by $F_i$ and, respectively, $\tilde{F}_i$, while those infinite pieces that are parallel openings at infinity we will mark with $^0$, and denote them as, respectively, $F_k^0$, $\tilde{F}_k^0$.

Since $F_i$ and $\tilde{F}_i$ satisfy the core assumptions, in particular assumption 3, each of them makes the contribution $\alpha_i := 2\varphi_i - \pi$, and, respectively, $\tilde{\alpha}_i := 2\tilde{\varphi}_i - \pi$ into $|\Phi(\mathcal{P})|$ and, respectively, $|\Phi(\tilde{\mathcal{P}})|$. But then the condition $|\Phi(\mathcal{P})| = |\Phi(\tilde{\mathcal{P}})|$ implies $\sum \alpha_i = \sum \tilde{\alpha}_i$. Such an equality allows us to assume that, with possibly some additional subdivisions along auxiliary rays, $\alpha_i = \tilde{\alpha}_i$ for each $i$, which is equivalent to $\varphi_i = \tilde{\varphi}_i$ for each $i$.

Therefore, we can always assure that $|\Phi(F_i)| = |\Phi(\tilde{F}_i)|$ for each $i$. In fact, this is enough to show $F_i \simeq \tilde{F}_i$. Let us show how to equidecompose such a pair of pieces.

First, if $|a_i| = |\tilde{a}_i|$, then $F_i$ and $\tilde{F}_i$ are automatically isometric. If $|a_i| \neq |\tilde{a}_i|$, then without loss of generality we may assume $a_i > \tilde{a}_i$. From the starting point of $\tilde{l}_i$ draw a ray $\tilde{r}_i'$ collinear to $\tilde{r}_i$. This ray will divide $\tilde{F}_i$ into two parts: a parallel opening at infinity $\tilde{S}_i$ and a solid angle $\tilde{F}_i'$. Since $a_i > \tilde{a}_i$, we can find an isometric copy $S_i$ of $\tilde{S}_i$ in $F_i$ such that one of its infinite edges is the ray $r_i$. Then $\overline{F_i \backslash S_i}$, which we will denote as $F_i'$, is a truncated solid angle whose boundary consists of a finite edge of length $|a_i| - |\tilde{a}_i|$ and two rays both making angle $\varphi_i$ with this edge. Note that

$$|\Phi(F_i')| = |\Phi(\tilde{F}_i')|. \tag{3.5}$$

Let us show that condition (3.5) implies $F_i' \simeq \tilde{F}_i'$. By construction, this will prove $F_i \simeq \tilde{F}_i$. Such implication follows from the next lemma.

**Lemma 1.** *Let $ABC$ and $\tilde{A}\tilde{B}\tilde{C}$ be two equal acute solid angles with vertexes at $B$ and $\tilde{B}$, and $K \in AB$, $L \in BC$ be two points such that $BK = BL$. Suppose $AKLC$ is the unbounded polygon obtained from $ABC$ by cutting out the triangle $KBL$; then $AKLC \simeq \tilde{A}\tilde{B}\tilde{C}$.*

*Proof.* Let $KM$ and $\tilde{K}\tilde{M}$ with $\tilde{K} \in \tilde{B}\tilde{A}$ be two rays lying inside the polygons $AKLC$ and $\tilde{A}\tilde{B}\tilde{C}$, respectively, such that $KM \parallel LC$, $\tilde{K}\tilde{M} \parallel \tilde{B}\tilde{C}$, and also such that the heights of the infinite strips $MKLC$ and $\tilde{M}\tilde{K}\tilde{B}\tilde{C}$ coincide. Pick two points $R \in KM$ and $S \in LC$ with the condition $RS \perp KM$. Choose the similar points $\tilde{R} \in \tilde{K}\tilde{M}$, $\tilde{S} \in \tilde{B}\tilde{C}$ with $\tilde{R}\tilde{S} \perp \tilde{K}\tilde{M}$. Obviously, we can take these four points in such a way that $Area(\tilde{K}\tilde{R}\tilde{S}\tilde{B}) = Area(KRSL)$. Finally, with the described construction we have obtained the following: for the solid angles

$\tilde{A}\tilde{K}\tilde{M} = AKM$, for the infinite strips $\tilde{M}\tilde{R}\tilde{S}\tilde{C} = MRSC$, and the trapezoids $\tilde{K}\tilde{R}\tilde{S}\tilde{B}$, $KRSL$ are equidecomposable by the classical Bolyai – Gerwien theorem. All this proves that $\tilde{A}\tilde{B}\tilde{C}$ and $AKLC$ are equidecomposable polygons. $\square$

From Lemma 1, $F_i' \simeq \tilde{F}_i'$, and thus $F_i \simeq \tilde{F}_i$.

So far we have shown how to equidecompose all unbounded pieces that make a non-trivial contribution to the limiting angles of the initial polygons. Let us now explain what to do with parallel openings at infinity $F_k^0$ and $\tilde{F}_k^0$ that may occur in polygons $\mathcal{P}$ and $\tilde{\mathcal{P}}$.

Suppose we have at least one $F_k^0 \subset \mathcal{P}$. For every $k$, choosing the segment $a_k^0$ appropriately we assume $\angle(a_k^0, r_k^0) = \tilde{\varphi}_1$, where $\tilde{\varphi}_1$ is the angle defined by assumption 3 for the first piece $\tilde{F}_1$. By possibly pushing the segment $\tilde{a}_1$ parallel to itself further toward infinity, we can make $|\tilde{a}_1|$ not less then $\sum_k |a_k^0|$. That allows us starting from the endpoint of the ray $\tilde{r}_1$ to mark on $\tilde{a}_1$ consecutive segments of lengths $|a_1^0|$, $|a_2^0|$, ... and so on. Finally, through each vertex of the obtained segments draw a ray collinear to $\tilde{r}_1$. These rays cut out of $\tilde{F}_1$ parallel openings at infinity $\tilde{S}_1^0$, $\tilde{S}_2^0$, ... isometrical to, respectively, $F_1^0$, $F_2^0$, ... and so on. Moreover, $\tilde{F}_{1,0} := \overline{F_1 \setminus \bigcup_k \tilde{S}_k^0}$ is an unbounded piece such that $|\Phi(\tilde{F}_{1,0})| = |\Phi(\tilde{F}_1)|$. Proceeding similarly in other way we can construct $F_{1,0}$ by cutting out of $F_1$ copies of all $\tilde{F}_k^0$ if there are any. This will get us $|\Phi(F_{1,0})| = |\Phi(F_1)| = |\Phi(\tilde{F}_1)| = |\Phi(\tilde{F}_{1,0})| \neq 0$, which by consideration above guaranties $F_{1,0} \simeq \tilde{F}_{1,0}$.

Therefore, we can equidecompose unbounded parts $\mathcal{F}$ and $\tilde{\mathcal{F}}$ both with parallel openings at infinity, or without. Together with the core assumption 4, which can be always fulfilled, this result shows the way how to equidecompose $\mathcal{P}$ and $\tilde{\mathcal{P}}$. Sufficiency is proved, and Theorem 1 with it.

# References


[1] V. G. Boltianskii, *Hilbert's Third Problem* (transl. R. A. Silverman), V. H. Winston & Sons (Halsted Press, John Wiley & Sons), Washington, D.C., 1978.

[2] A. M. Petrunin, S. E. Rukshin, *Уникальносоставленные фигуры*, Mat. Pros., Ser. 3, **10**, MCCME, Moscow, 2006, 161–175.

[3] A. D. Alexandrov, *Convex Polyhedra*, Springer-Verlag, Berlin, 2005.