Solution of the initial value problem for the inhomogeneous equation of vibrations of a finite string with homogeneous boundary conditions

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One of the most known methods of solution of the one-dimensional wave equation is the method of traveling waves. Its main idea with reference to the first initial boundary value problem for a finite string with fixed ends consist in the following. One should construct the odd periodic continuation of initial functions to overall real axis and then substitute the obtained functions into d'Alembert formula. But it is difficult to express by a unified formula a disturbance reflected back and forth between the ends of the string. In the paper [1] we give such unified equation for the free vibrations problem of a homogeneous string with the fixed end points. Here we give a solution of the forced oscillation problem of the string.

Let on the segment $[0, L]$ the one-dimensional wave equation is given $u_{tt} = a^2 u_{xx} + f(x,t)$. Let the initial conditions $u(x,0) = \varphi(x)$, $u'_t(x,0) = \psi(x)$ and the homogeneous boundary conditions $u(0,t) = 0$, $u(L,t) = 0$, $t \geq 0$ are given. The solution of this initial boundary value problem can be written

$$u(x,t) = \frac{1}{2}\Big((-1)^{\left[\frac{x+at}{L}\right]}\varphi(stc(x+at,2L)) + (-1)^{\left[\frac{x-at}{L}\right]}\varphi(stc(x-at,2L))\Big) +$$

$$+\frac{1}{2a}\int\limits_{stc(x-at,2L)}^{stc(x+at,2L)} \psi(\alpha)d\alpha + \frac{1}{2a}\int\limits_0^t d\tau \int\limits_{stc(x-a(t-\tau),2L)}^{stc(x+a(t-\tau),2L)} f(\xi,t)d\xi \;, \tag{1}$$

where $[z]$ gives the greatest integer less than or equal to z and expression $stc(x,w)$ represents a sawtooth function defined by the formula

$$stc(x,w) = \frac{w}{2\pi}\arccos\cos\frac{2\pi x}{w} \equiv \left| x - w\left[\frac{x}{w}+\frac{1}{2}\right]\right|$$

Equation (1) represents an explicit solution of the inhomogeneous equation of vibrations of a finite string with the fixed ends. If the functions $f \in C^1$, $\varphi \in C^2$, $\psi \in C^1$ satisfy to known consistency constraints

$$\varphi(0) = \varphi(L) = 0, \psi(0) = \psi(L) = 0, \varphi''(0)\,a^2 + f(0,0) = \varphi''(L)\,a^2 + f(L,0) = 0$$

then the solution, given by the formula (1), will have continuous derivatives of the first and the second orders.

[1] P.G. Dolya, *Periodic continuation of functions and solution of the string vibration equation in systems of symbol mathematics.* Bulletin of V.Karazin Kharkiv National University. Series "Mathematical Modelling. Information Technology. Automated Control Systems", **733**, v.6 (2006), p. 106–116 (in Russian).