Geometry of hypersurfaces with bounded normal curvatures

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(joint work with prof. Alexander Borisenko[1])

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If $\gamma(s) \subset M$ is the normal curve s.t. $\gamma(0) = x$, $\dot\gamma(0) = u$, then

$$\kappa_{\mathrm{n}}^N(x,u) = \langle \nabla_{\dot\gamma}\dot\gamma, N\rangle$$

– projection of the curvature vector of γ on N.

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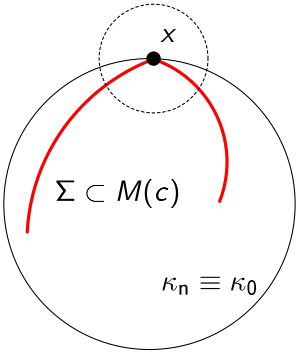
- Main object #2:
  Complete κ_0-convex $\Sigma \subset M(c)$ (non-smooth case)

κ_0-convexity: a picture

$\Sigma \subset M(c)$ is κ_0-convex if for any $x \in \Sigma$



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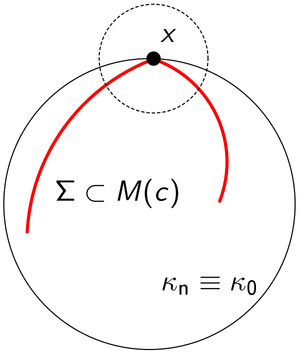

- For $M(c) = \mathbb{H}(-k^2)$ and $\kappa_0 = k$ we get horospherical convexity or h-convexity.

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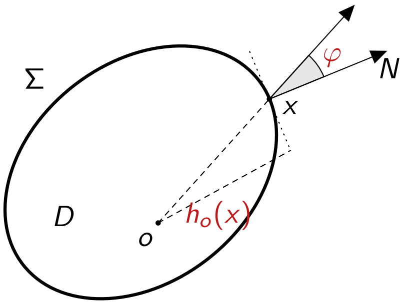

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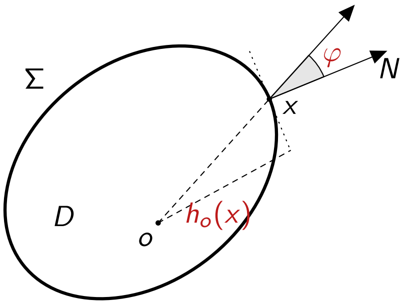

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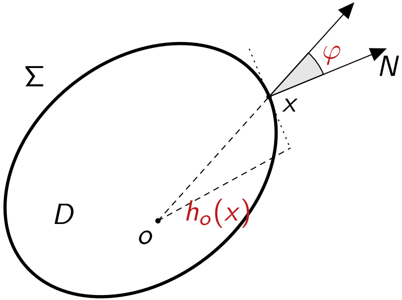

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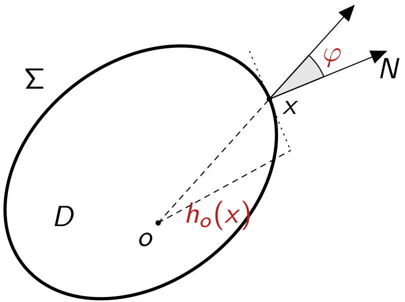

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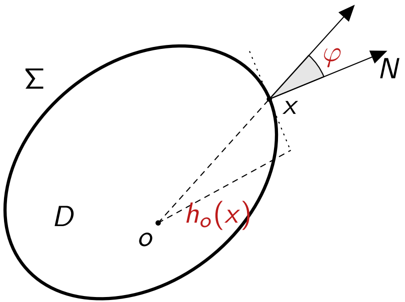

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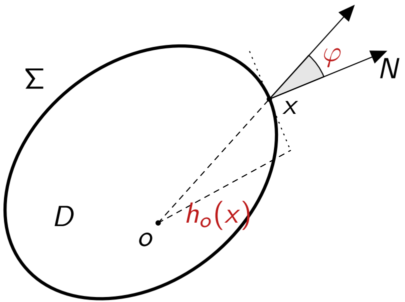

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$h_o(x) = \rho(o, x) \cdot \cos\varphi$ – *Riemannian support function* (*where $\rho(o, x)$ is the distance between o and x*)

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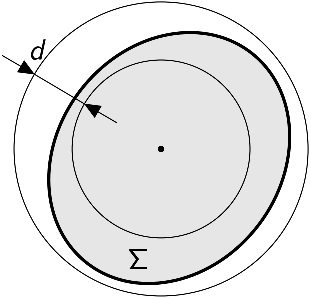

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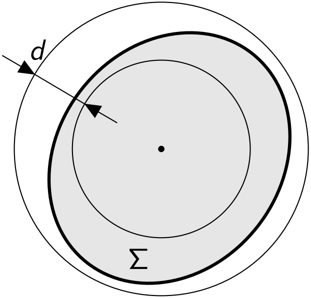

Width of a shell between two concentric spheres that encloses Σ

$$d = R - r$$

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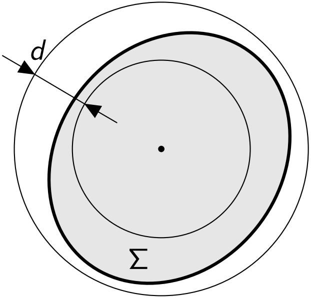

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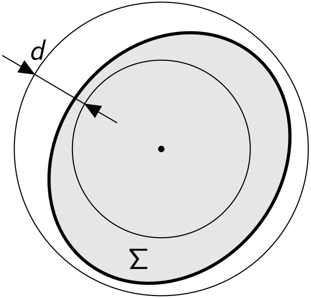

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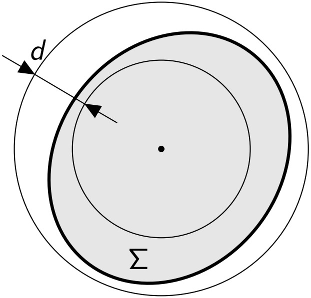

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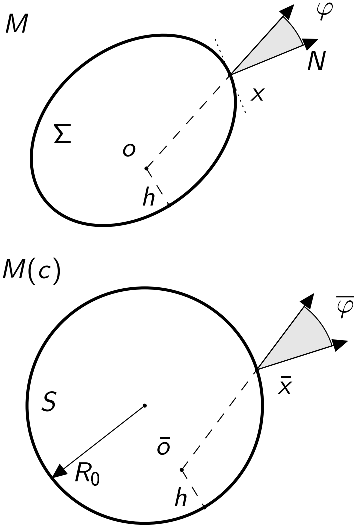
- Generalization of the classical Fenchel and Fary – Milnor inequalities in Minkowski geometry *(A. Borisenko, K. Tenenblat, 2012)*

Radial Angle Comparison Theorem



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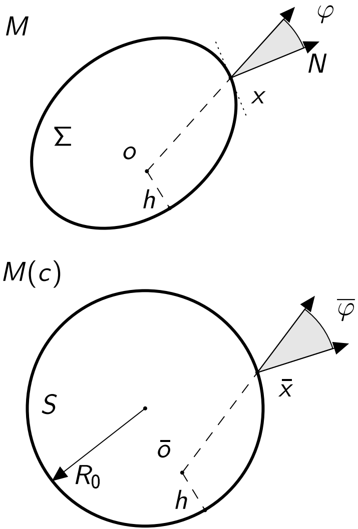

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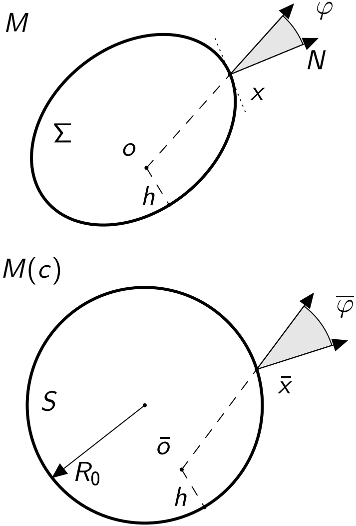

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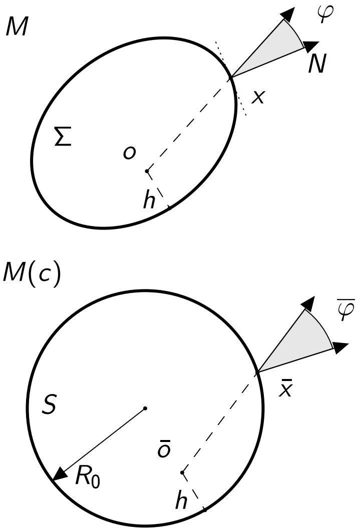

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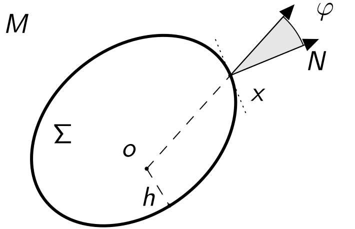

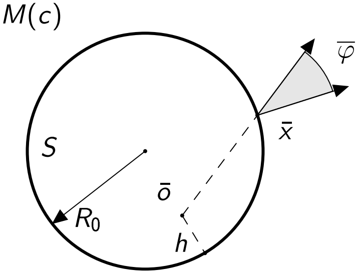

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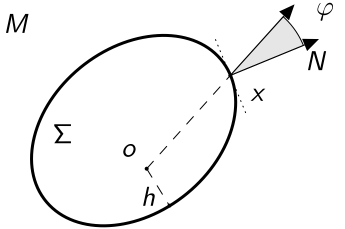

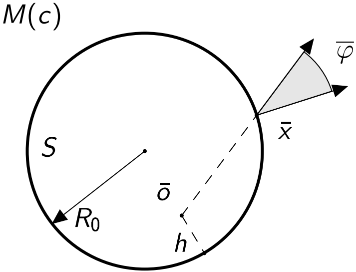

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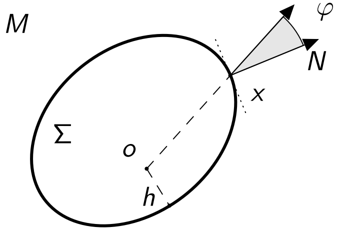

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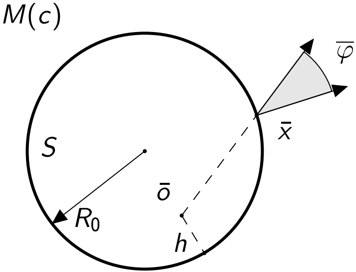

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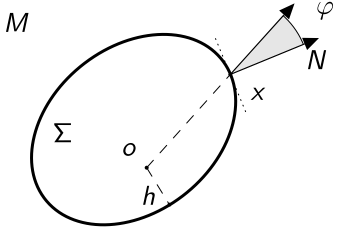

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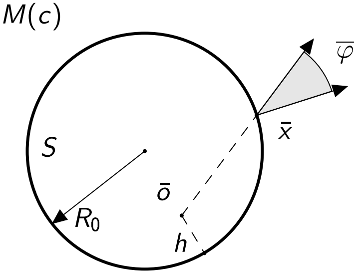

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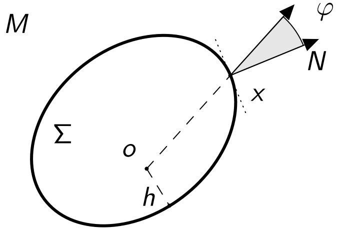

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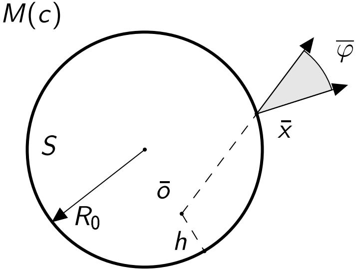

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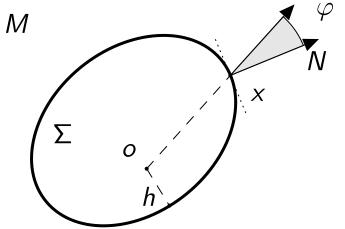

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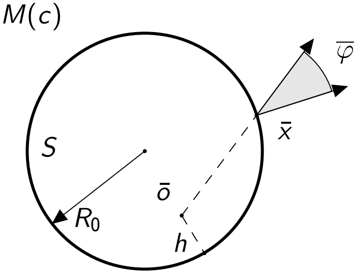

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and let $\overline{\varphi}(\bar{x})$ be the corresponding radial angle at $\bar{x}$.

Radial Angle Comparison Theorem

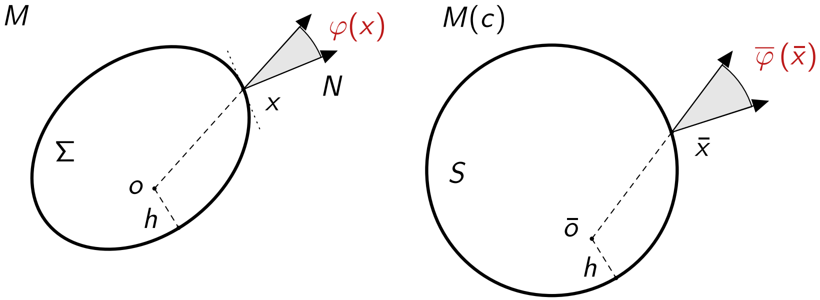
Theorem (A. Borisenko, K Drach)

In the notations above

$$\varphi(x) \leqslant \overline{\varphi}(\bar{x})$$

and (for support functions)

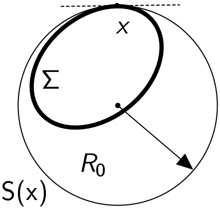
$$h_o(x) \geqslant h_{\bar{o}}(\bar{x})$$



Radial Angle Comparison Theorem: application

Theorem (Generalized Blaschke's Rolling Theorem)

Let $\Sigma \subset M(c)$ be a complete κ_0-convex hypersurface (where for $c = -k^2 < 0$, $\kappa_0 > k$).
Then for every point $x \in \Sigma$ the hypersurface Σ is contained inside the corresponding supporting sphere $S(x)$ of radius R_0 passing through x.



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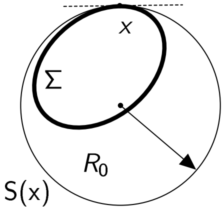

Similar holds for κ_0-concave hypersurfaces.

Radial angle bounds

Theorem (A. Borisenko, K. Drach)

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3. *If $c = -k^2 < 0$ and $\kappa_0 > k$, then*

$$\cos\varphi \geqslant \frac{\sinh kh}{\sinh kR_0}. \tag{3}$$

Radial angle bounds

Theorem (A. Borisenko, K. Drach)

Let M be a complete simply connected Riemannian manifold, $\Sigma \subset M$.

1. *Let all sectional curvatures $0 \geqslant K_\sigma \geqslant -k^2$, $k > 0$. If $\kappa_{\mathrm{n}} \geqslant \kappa_0 > k$, then estimate (3) holds.*

$$\cos\varphi \geqslant \frac{\sinh kh}{\sinh kR_0}$$

2. *Let sectional curvatures $k_1^2 \geqslant K_\sigma \geqslant k^2 > 0$, and Σ lies in a ball of radius $\pi/2k_1$. If $\kappa_{\mathrm{n}} \geqslant \kappa_0 \geqslant 0$, then (1) holds.*

$$\cos\varphi \geqslant \frac{\sin kh}{\sin kR_0}$$

Radial Angle Comparison Theorem: glimpse on the proof

Theorem (A. Borisenko)

Let $\Sigma \subset M$ be a smooth hypersurface, N be its unit (outward pointing) Gauss map, $o \in M \backslash \Sigma$ be a fixed point, ρ be the distance function from o, and φ be the corresponding radial angle of Σ w.r.t. o. Suppose $\rho_\Sigma = \rho|_\Sigma$. For any $x \in \Sigma$ let $\gamma(s)$ be an arc-length parameterized integral trajectory of $\operatorname{grad}_\Sigma \rho_\Sigma$ passing through $x = \gamma(0)$. Then along γ

$$\kappa_{\mathrm{n}}^{N}(\gamma(s), \operatorname{grad}_\Sigma \rho_\Sigma) = \mu_{\mathrm{n}}^{\operatorname{grad} \rho}(\gamma(s), u) \cdot \cos\varphi - \frac{d\varphi}{ds}, \tag{4}$$

where $\mu_{\mathrm{n}}^{\operatorname{grad} \rho}(\gamma(s), u)$ is the normal curvature of the geodesic sphere with center o and radius $\rho(0, \gamma(s))$ in the direction of u with u being a vector in the 2-plane spanned by N and $\operatorname{grad} \rho$ orthogonal to the later.

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$$d \leqslant \left(\sqrt{2} - 1\right)(R_0 - R_1); \tag{5}$$

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$$d \leqslant \frac{2}{k} \arccos \sqrt{\cos\left(k(R_0 - R_1)\right) - (R_0 - R_1)}; \tag{6}$$

3. *for $c = -k^2$, $k > 0$, and $\kappa_0 > k$,*

$$d \leqslant \frac{2}{k} \operatorname{arccosh} \sqrt{\cosh\left(k(R_0 - R_1)\right) - (R_0 - R_1)}. \tag{7}$$

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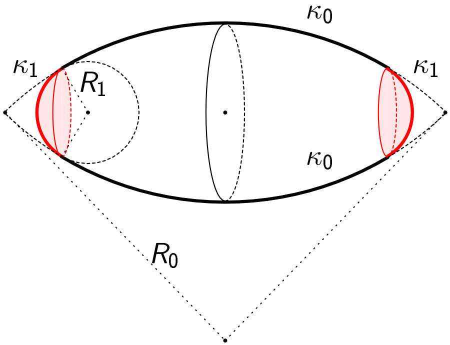
Moreover, these estimates are sharp.

Bounds on the width: extreme surface

- The equality holds for rounded spindle-shaped hypersurface



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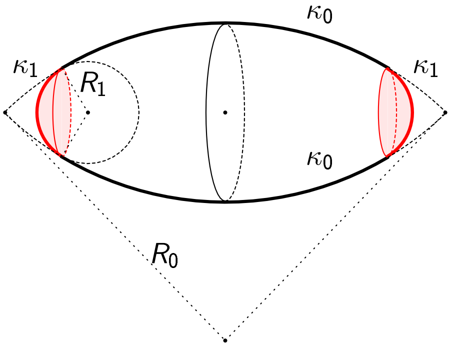

- For $M^2(c)$ the equality holds only for rounded spindle-shaped hypersurfaces – rounded lunes.

Bounds on the width for κ_0-convexity

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Corollary (A. Borisenko, K. Drach)

Let $\Sigma \subset M(c)$ be a complete κ_0-convex hypersurface.

1. *If $c = 0$, then*
$$d \leqslant (\sqrt{2} - 1)R_0. \tag{8}$$
2. *If $c = k^2$, then*
$$d \leqslant \frac{2}{k} \arccos \sqrt{\cos kR_0} - R_0 \tag{9}$$
3. *If $c = -k^2$ and $\kappa_0 > k$, then*
$$d \leqslant \frac{2}{k} \operatorname{arccosh} \sqrt{\cosh kR_0} - R_0 \tag{10}$$

Moreover, these bounds are sharp.

Bounds on the width for κ_0-convexity (continued)

Using the Radial Angle Comparison Theorem we can also get

Theorem (A. Borisenko, K. Drach)

$\Sigma \subset M$ be a complete smooth hypersurface with $\kappa_n \geqslant \kappa_0 > 0$.

1. *If M is a Hadamard manifold with sectional curvatures $0 > K_\sigma \geqslant -k^2$, $k > 0$, and $\kappa_0 > k$, then the width d admits the estimate (10)*

$$d \leqslant \frac{2}{k} \operatorname{arccosh} \sqrt{\cosh kR_0} - R_0.$$

2. *Assume that sectional curvatures of M are bounded $k_1^2 \geqslant K_\sigma \geqslant k^2 > 0$, and suppose Σ lies in a ball of radius π/k_1 which center that coincides with the center of the inscribed ball for Σ. Then the width d admits the estimate (9)*

$$d \leqslant \frac{2}{k} \arccos \sqrt{\cos kR_0} - R_0.$$

Bounds on the quotient of a shell's radii: piece of motivation

$F^2 \subset \mathbb{E}^3$ – closed convex surface, k_1, k_2 – principal curvatures

If $\frac{k_1}{k_2} = 1$ at every point, then F^2 – sphere;

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If pointwise on F^2

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For κ_0-convex hypersurface there are examples with the arbitrary large quotient R/r!

Bounds on the quotient of a shell's radii for κ_0, κ_1-convexity

Theorem (K. Drach)

Let $\Sigma \subset \mathbb{E}^{m+1}$ be a complete κ_0, κ_1-convex hypersurface in the Euclidean space. Then the hypersurface Σ can be put into a spherical shell between two concentric spheres of radii r and R ($R \geqslant r$) such that

$$\frac{R}{r} \leqslant \frac{\sqrt{\frac{\kappa_1}{\kappa_0}} + \sqrt{2}}{\sqrt{\frac{\kappa_0}{\kappa_1}} + \sqrt{2}}. \tag{11}$$

Moreover, this estimate is sharp.

- The equality holds for rounded spindle-shaped hypersurfaces (for $m = 1$, only for them).