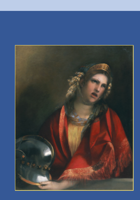
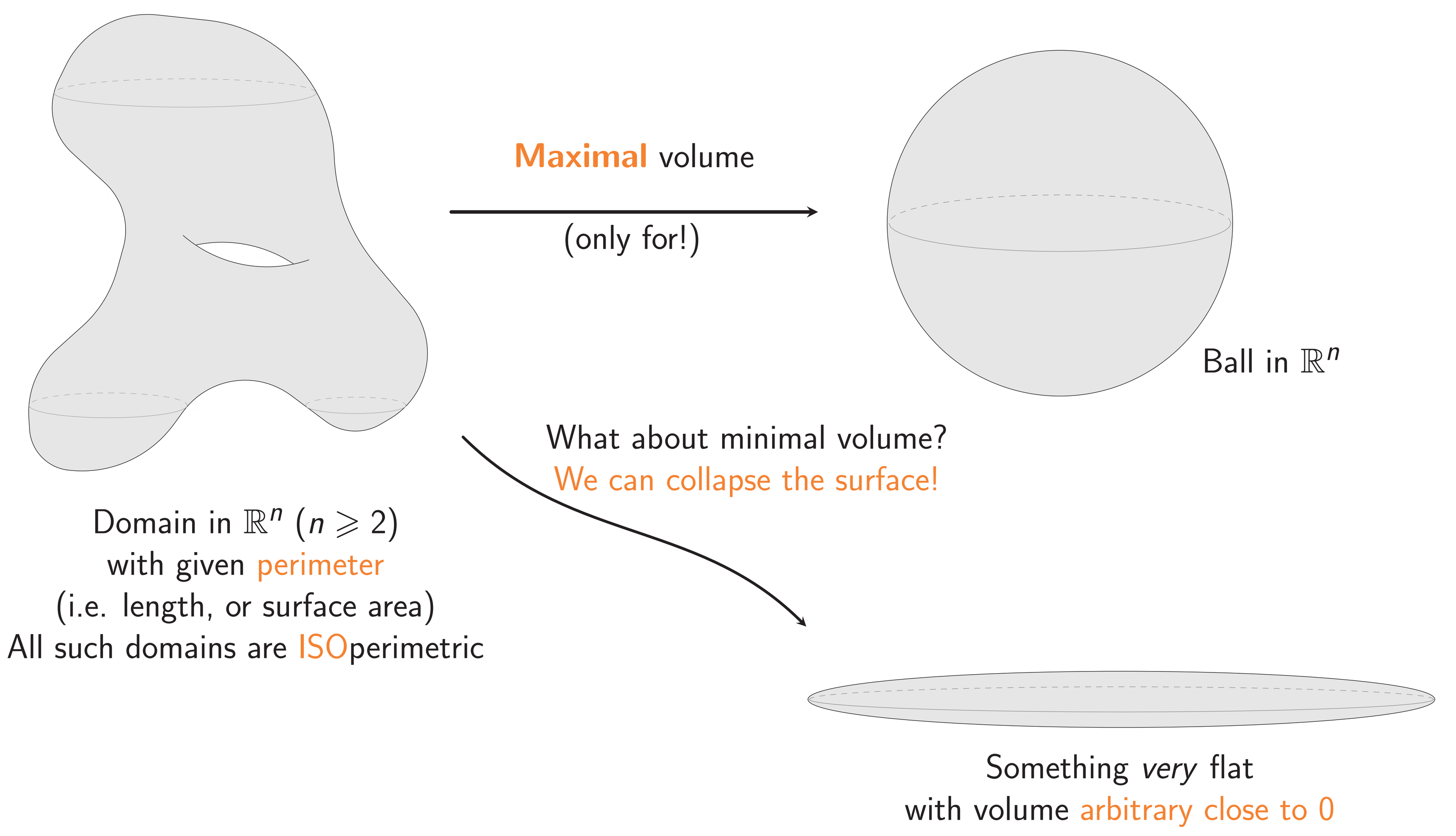
The Reverse Isoperimetric Inequalities for Strictly Convex Domains

Kostiantyn Drach
(joint work with prof. A. Borisenko)
V.N. Karazin Kharkiv National University, Sumy State University (Ukraine)

The classical isoperimetric inequality (Dido's Problem)



Queen Dido



Trivial case!

Precise statements

Theorem (Solution of Dido's Problem)

Suppose a closed surface $\Sigma \subset \mathbb{R}^n$ and a sphere $\mathcal{S} \subset \mathbb{R}^n$ have equal perimeters (length/surface areas)

$$\text{Perimeter}(\Sigma) = \text{Perimeter}(\mathcal{S}),$$

then

$$\text{Volume}(\Sigma) \leqslant \text{Volume}(\mathcal{S}),$$

where Volume$(\cdot)$ *is the volume of the domain enclosed by the corresponding surface. Moreover, equality holds only if $\Sigma = \mathcal{S}$.*

Equivalent statement for $\mathbb{R}^2/\mathbb{R}^3$.

Theorem (Isoperimetric inequality)

($\mathbb{R}^2$) *Let L and A be the length of the boundary and the area of a plane domain; then*

$$4\pi A \leqslant L^2,$$

and equality holds only if the domain is a disc.
($\mathbb{R}^3$) *Let A and V be the surface area of the boundary and the volume of a domain in $\mathbb{R}^3$; then*

$$36\pi V^2 \leqslant A^3,$$

with equality only if the domain is a ball.

Reversing and what is known about it?

Restricting **curvature** of the boundary of a domain, we can obtain a **non-trivial reverse isoperimetric inequality**.
In [6], [5] authors investigated the reverse isoperimetric problem for curves in $\mathbb{R}^2$ ([6]) and, partially, surfaces in $\mathbb{R}^3$ ([5]) with curvatures $|k| \leqslant 1$ (in a weak sense). Such a bound allows curves and surfaces to be non-convex, and for such classes the reverse isoperimetric problem is not well-defined.

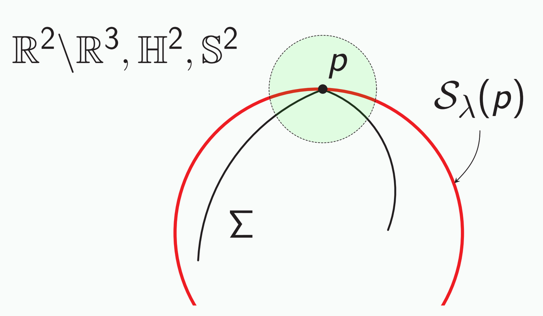
We consider convex curves and surfaces with restricted curvature.

What do we mean by restricting curvature?

For simplicity, we focus only on the objects in the spaces where the results are known so far. That is, we will consider *curves in 2-dimensional constant curvature spaces* — hyperbolic space $\mathbb{H}^2$ of curvature -1, Euclidean plane $\mathbb{R}^2$ of curvature 0, sphere $\mathbb{S}^2$ of curvature 1, — and *surfaces in* $\mathbb{R}^3$.

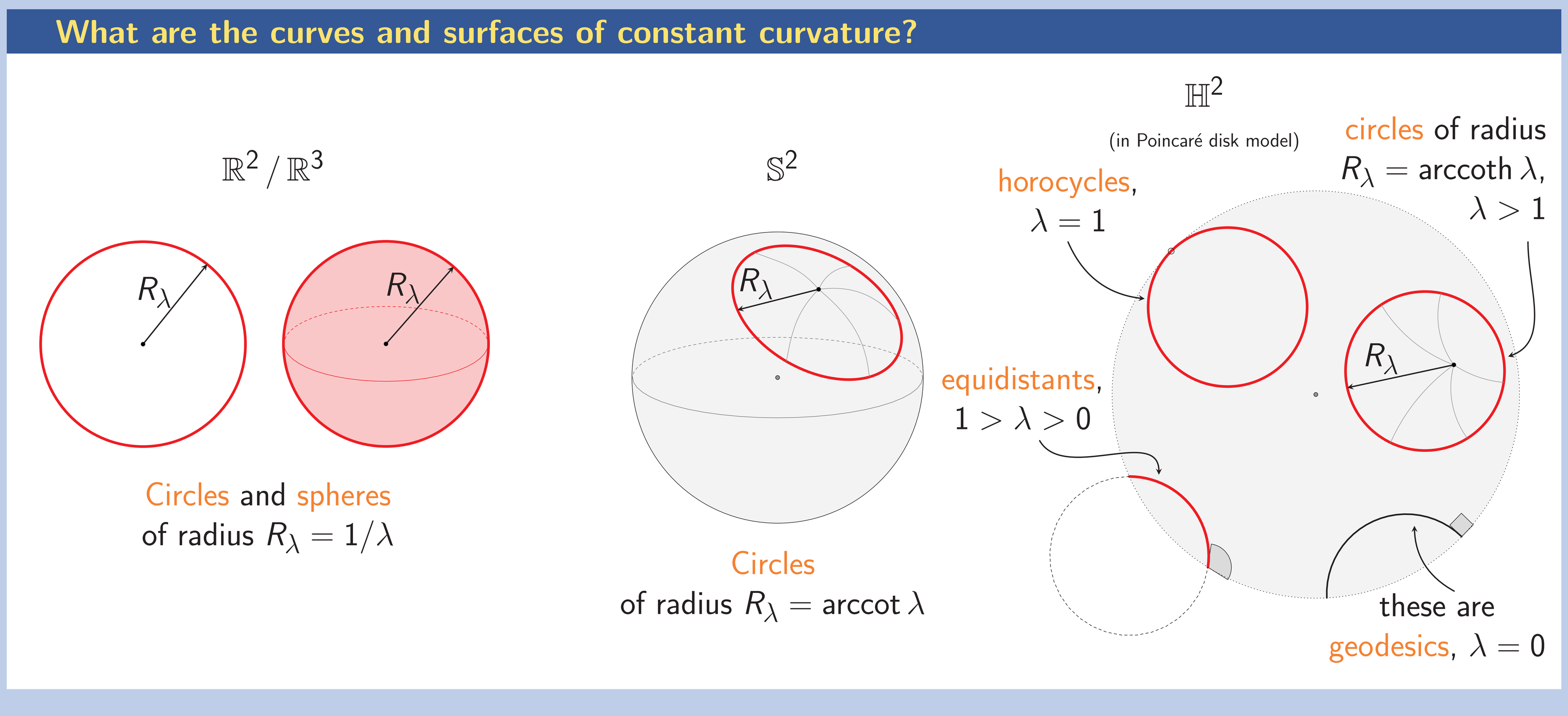
Definition of λ-convexity

A locally convex hypersurface Σ is called λ-convex (with $\lambda > 0$) if for any point $p \in \Sigma$ there exists a curve/surface $\mathcal{S}_\lambda(p)$ of constant curvature λ passing through p in such a way, that in some neighborhood of p the surface Σ lies inside $\mathcal{S}_\lambda(p)$.
λ-convex domain, by definition, is a domain with λ-convex boundary.



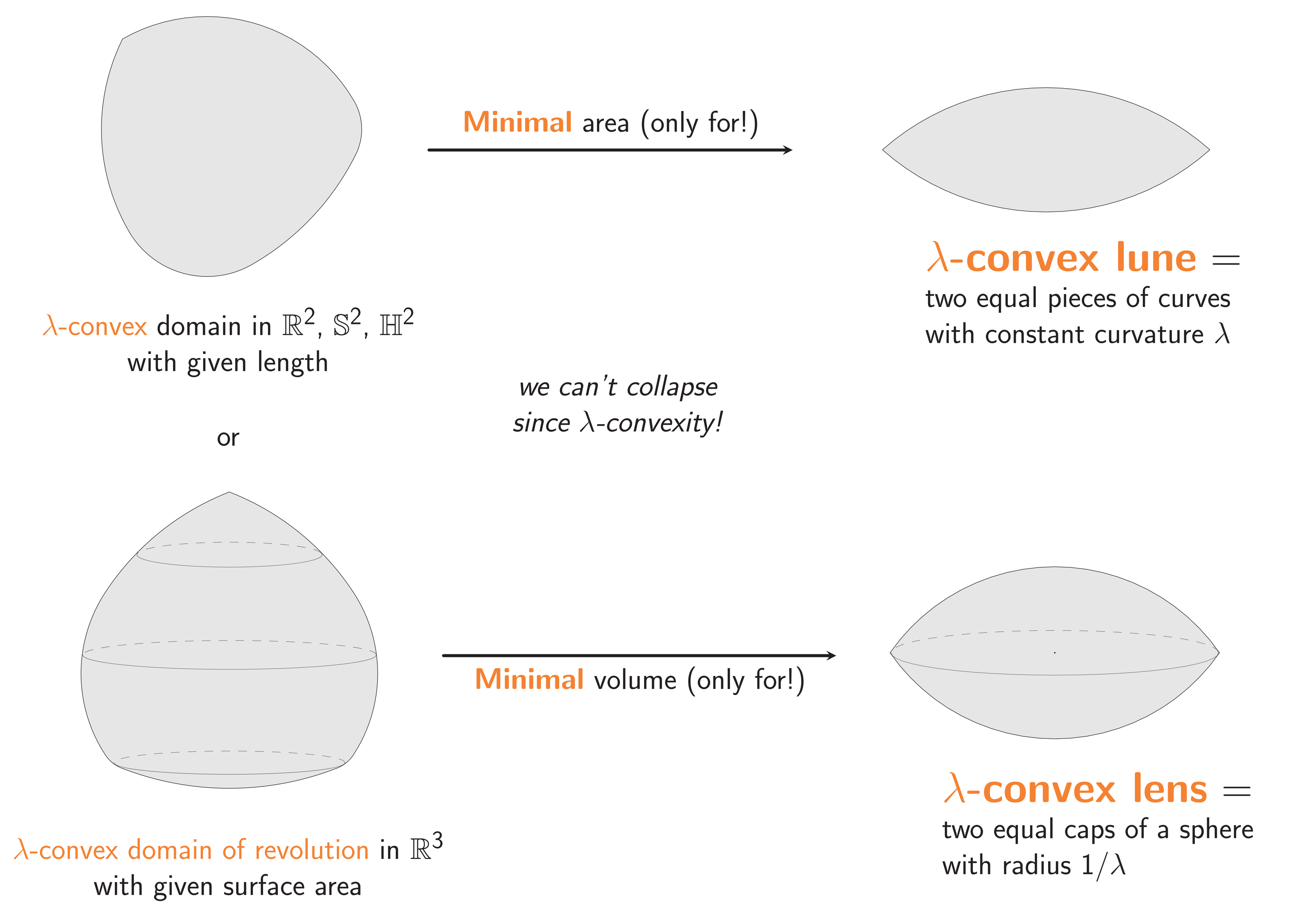
Informally: a curve/surface is λ-convex if it bends *not slower* than the curve/surface of constant curvature λ.
In the smooth case: if Σ is *smooth* λ-convex curve, then it's curvature at each point satisfies $k \geqslant \lambda$ (in the case of surfaces, k is the *normal curvature*).

What are the curves and surfaces of constant curvature?



The reverse isoperimetric inequality (Reverse Dido's Problem)

Queen Dido (mirrored)



Precise statements

Theorem ([1, 3, 2], Solution of Reverse Dido's Problem in 2-dim constant curvature spaces)

Let Σ be a closed embedded λ-convex curve lying in $\mathbb{R}^2$, $\mathbb{S}^2$, or $\mathbb{H}^2$. If Σ_λ is a λ-convex lune such that

$$\text{Perimeter}(\Sigma) = \text{Perimeter}(\Sigma_\lambda),$$

then

$$\text{Volume}(\Sigma) \geqslant \text{Volume}(\Sigma_\lambda).$$

Moreover, equality holds if and only if Σ is a λ-convex lune.

Theorem ([4], Partial solution of the Reverse Dido's Problem in $\mathbb{R}^3$)

Let $\Sigma \subset \mathbb{R}^3$ be a complete λ-convex surface of revolution. If $\Sigma_\lambda \subset \mathbb{R}^3$ is a λ-convex lens such that

$$\text{Perimeter}(\Sigma) = \text{Perimeter}(\Sigma_\lambda),$$

then

$$\text{Volume}(\Sigma) \geqslant \text{Volume}(\Sigma_\lambda),$$

and equality is attained if and only if Σ is congruent to Σ_λ.

The theorems above are equivalent to the following.

Theorem (Reverse isoperimetric inequality)

($\mathbb{R}^2, \mathbb{S}^2, \mathbb{H}^2$) *Let Σ be a closed embedded λ-convex curve of length L and enclosing a domain of area A.*

1. *If $\Sigma \subset \mathbb{R}^2$, then*
$$A \geqslant \frac{L}{2\lambda} - \frac{1}{\lambda^2}\sin\frac{L\lambda}{2}.$$
2. *If $\Sigma \subset \mathbb{S}^2$, then*
$$A \geqslant 4\arctan\left(\frac{\lambda}{\sqrt{\lambda^2+1}}\tan\left(\frac{\sqrt{\lambda^2+1}}{4}L\right)\right) - \lambda L.$$
3. *If $\Sigma \subset \mathbb{H}$, and $\lambda > 1$, then*
$$A \geqslant \lambda L - 4\arctan\left(\frac{\lambda}{\sqrt{\lambda^2-1}}\tan\left(\frac{\sqrt{\lambda^2-1}}{4}L\right)\right).$$
4. *If $\Sigma \subset \mathbb{H}$, and $\lambda = 1$, then*
$$A \geqslant L - 4\arctan\frac{L}{4}.$$
5. *If $\Sigma \subset \mathbb{H}$, and $1 > \lambda > 0$, then*
$$A \geqslant \lambda L - 4\arctan\left(\frac{\lambda}{\sqrt{1-\lambda^2}}\tanh\left(\frac{\sqrt{1-\lambda^2}}{4}L(\gamma)\right)\right).$$

Equality holds only for a λ-convex lune.
($\mathbb{R}^3$, **rotational case**) *Suppose $\Sigma \subset \mathbb{R}^3$ is a complete λ-convex surface of revolution, V and A are the volume of the domain enclosed by Σ and the surface area of Σ, respectively; then*

$$96\pi^2 V \geqslant \lambda A^2\left(12\pi - \lambda^2 A\right),$$

and equality holds if and only if Σ is a λ-convex lens.

Open problems

The result in $\mathbb{R}^3$ above is the only known result on the solution of Reverse Dido's Problem in higher dimensions so far. The following conjecture is very plausible.

Conjecture (due to A. Borisenko)

For the spaces of constant curvature the solution for the analogous multi-dimensional reverse isoperimetric problem in the class of complete λ-convex hypersurfaces is a convex joint of two equal totally umbilical caps of normal curvature equal to λ (lens hypersurface).

References

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